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Existence and multiplicity of periodic solutions for a class of second-order Hamiltonian systems☆

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ABSTRACT

We study the existence and multiplicity of periodic solutions of the following second-order Hamiltonian system

$$\ddot{u}(t) + \nabla[-K(t, u(t)) + W(t, u(t))] = 0.$$

The existence of a nontrivial periodic solution is obtained when ∇W is asymptotically linear at infinity, and the existence of infinitely many periodic solutions is also obtained when ∇W is superlinear.

1. Introduction and main results

Consider the second-order Hamiltonian systems

$$\ddot{u}(t) + \nabla F(t, u(t)) = 0, \tag{1.1}$$

where $t \in \mathbb{R}$, $u \in \mathbb{R}^N$ and $F : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies:

(F1) $F(t, x) = -K(t, x) + W(t, x)$, $K, W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and are T-periodic in its first variable with $T > 0$.

Recently the existence and multiplicity of periodic solutions for system (1.1) have been studied in many papers via critical point theory; see [1–20]. In a pioneering work, Rabinowitz [13] established the existence of periodic solutions for system (1.1) under the well known Ambrosetti–Rabinowitz condition: there exist some $\mu > 2$ such that

$$0 < \mu F(t, x) \le (\nabla F(t, x), x) \tag{AR}$$

for all $t \in [0, T]$ and $x \in \mathbb{R}^N \setminus \{0\}$. Since then, the (AR)-condition has been used extensively to study superlinear Hamiltonian systems, for example, see [21–23,11,20] and references therein. Under the usual (AR)-condition, it is easy to show that the energy functional associated with the system has the Mountain Pass geometry and satisfies the (PS)-condition. Recently, Fei [6] and Tao and Tang [16] studied the existence of periodic solutions for Hamiltonian systems under a different superlinear conditions.

However, since (AR) does not hold for asymptotically linear cases, the problem becomes more delicate and hence some topology tools are involved. Using a Morse theory for strongly indefinite functionals, Abbondandolo [1] obtained periodic solutions for a class of first-order systems which are T-resonant at infinity. Fei [7] obtained, via a Maslov-type index, a non-trivial periodic solution for first-order systems which are resonant at infinity; the results were generalized later by Su [14].

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See also Fei [8] for multiple results. First-order systems which are resonant both at zero and at infinity were considered in Szulkin and Zou [15], Guo [9], Fei [5]. Degiovanni and Olian Fannio [4] proved multiple periodic solutions for autonomous asymptotically linear first-order Hamiltonian systems, on the basis of spectral properties of the matrices $H_{zz}(\infty)$ and $H_{zz}(0)$, where H is the Hamiltonian of the system and H_{zz} denotes its Hessian.

In recent paper [19], Zhao, Chen and Yang studied the existence of periodic solutions of system (1.1) with asymptotically linear function $\nabla F(t,x)$. Unlike in the works mentioned above, the behavior of $\nabla F(t,x)$ at infinity is like that of a function $V_\infty(t)x$, where $V_\infty(t)$ is a real valued function but not a matrix valued function. In detail, they obtained the following theorem.

Theorem A (*[19]*). *Assume that F satisfies* (F1), *and that K and W satisfy the following conditions:*

(K1) *There exist two constants $b_1 > 0$ and $b_2 > 0$ such that for all $(t,x) \in [0,T] \times \mathbb{R}^N$*

$$b_1|x|^2 \le K(t,x) \le b_2|x|^2;$$

(K2) *$K(t,x) \le (x, \nabla K(t,x)) \le 2K(t,x)$ for all $(t,x) \in [0,T] \times \mathbb{R}^N$;*

(W1) *$\nabla W(t,x) = o(|x|)$, as $|x| \to 0$ uniformly for $t \in [0,T]$;*

(W2) *There exists a function $V_\infty \in L^\infty([0,T],\mathbb{R})$ such that*

$$\lim_{|x|\to\infty} \frac{|\nabla W(t,x) - V_\infty(t)x|}{|x|} = 0 \quad \text{uniformly for } t \in [0,T]$$

and

$$\inf_{t\in[0,T]} V_\infty(t) > \frac{4\pi^2 + T^2}{T^2} \max\{1, 2b_2\};$$

(W3) *$\lim_{|x|\to\infty}[(\nabla W(t,x),x) - 2W(t,x)] = +\infty$ uniformly for $t \in [0,T]$.*

Then system (1.1) has a nontrivial T-periodic solution.

In view of the proof of [19, Theorem 1.1] (see the proof [19, Lemma 3.1]), the condition (W3) should be stated as in Theorem A instead of as in [19, Theorem 1.1]. Motivated by papers [10,19], in this paper, we will further study the existence of T-periodic solutions of (1.1) under more general conditions. Our first result is the following theorem.

Theorem 1.1. *Assume that F satisfies* (F1), *and that K and W satisfy*

(K1′) *There exist constants $b > 0$ and $\gamma \in (1,2]$ such that*

$$K(t,0) = 0, \qquad K(t,x) \ge b|x|^\gamma \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}^N;$$

(K2′) *$(\nabla K(t,x),x) \le 2K(t,x)$ for $(t,x) \in [0,T] \times \mathbb{R}^N$;*

(W1′) *$\limsup_{|x|\to 0} \frac{W(t,x)}{|x|^2} < b$ uniformly for $t \in [0,T]$;*

(W3′) *There exists a function $g \in L^1([0,T],\mathbb{R})$ such that*

$$(\nabla W(t,x),x) - 2W(t,x) \ge g(t) \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}^N$$

and

$$\lim_{|x|\to\infty}[(\nabla W(t,x),x) - 2W(t,x)] = +\infty \quad \text{for a.e. } t \in [0,T];$$

(W4) *There exist constants $a > 0$ and $d > 0$ such that*

$$W(t,x) \le a|x|^2 + d \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}^N;$$

(W5) *There exists $x_0 \in \mathbb{R}^N$ such that*

$$\int_0^T \left[K(t,x_0) - W(t,x_0) - \frac{g(t)}{2}\right] \mathrm{d}t < 0.$$

Then system (1.1) has a nontrivial T-periodic solution.

Corollary 1.1. *Assume that F, K and W satisfy* (F1), (K1′), (K2′), (W1′), (W3′) *and*

(W2′) *There exists a function $V_\infty \in L^\infty([0,T],\mathbb{R})$ such that*

$$\lim_{|x|\to\infty} \frac{|\nabla W(t,x) - V_\infty(t)x|}{|x|} = 0 \quad \text{uniformly for } t \in [0,T]$$

and

$$\int_0^T \left[\max_{|x|=1} K(t,x) - \frac{V_\infty(t)}{2}\right] \mathrm{d}t < 0.$$

Then system (1.1) has a nontrivial T-periodic solution.

Remark 1.1. If $K(t,x) \le b_2|x|^2$ for all $(t,x) \in [0,T]\times\mathbb{R}^N$ and $\inf_{t\in[0,T]} V_\infty(t) > 2b_2$, then the second inequality in (W2′) holds. This shows (W2′) is weaker than (W2). In addition, it is easy to see that (K1′), (K2′), (W1′) and (W3′) are weaker than (K1), (K2), (W1) and (W3), respectively. Therefore, Corollary 1.1 improves Theorem A.

Remark 1.2. (K1) is the so-called "pinching" condition. Except $K(t,x) = \frac{1}{2}(L(t)x,x)$, where $L(t)$ is a continuous symmetric and positive definite matrix valued function for all $t\in[0,T]$, as pointed in [23], there are still other functions $K(t,x)$ satisfy conditions (K1) and (K2). However, the conditions (K1) and (K2) are indeed rather harsh, there are much functions $K(t,x)$ which satisfy our conditions (K1′) and (K2′) but not the conditions (K1) and (K2). For example,

$$K(t,x) = b_1|x|^\gamma + b_2|x|^\varrho,$$

where $b_1, b_2 > 0$ and $1 < \gamma < \varrho \le 2$.

Remark 1.3. There are functions W satisfying (W3′) and not satisfying (W3). For example, let $T = 2\pi$ and

$$W(t,x) = |x|^2 + |x|^2\int_1^{|x|}\left[1+\sin\left(\frac{3\pi s}{1+s}-\frac{3t}{4}\right)\right]\mathrm{d}s. \tag{1.2}$$

Then we have

$$\tilde{W}(t,x) := (\nabla W(t,x),x) - 2W(t,x) = \left[1+\sin\left(\frac{3\pi|x|}{1+|x|}-\frac{3t}{4}\right)\right]|x|^3.$$

For every $t\in[0,2\pi]$, it is easy to verify that

$$\lim_{|x|\to\infty}\tilde{W}(t,x) = \lim_{|x|\to\infty}\left[1+\sin\left(\frac{3\pi|x|}{1+|x|}-\frac{3t}{4}\right)\right]|x|^3 = +\infty,$$

but for every $x\in\mathbb{R}^N$ with $|x|\ge 1$, there exists $t = 2\pi(|x|-1)/(|x|+1) \in [0,2\pi]$ such that $\tilde{W}(t,x) = 0$, and so $\lim_{|x|\to\infty}[(\nabla W(t,x),x) - 2W(t,x)] = \lim_{|x|\to\infty}\tilde{W}(t,x) = +\infty$ not uniformly for $t\in[0,2\pi]$.

Motivated by papers [6,16], we establish the following second theorem under the superquadratic condition which was given by Fei [6], see also Tao and Tang [16].

Theorem 1.2. *Assume that F, K and W satisfy* (F1), (K1′), (K2′), (W1′) *and*

(W4′) *There exist $a, d > 0$ and $\nu \ge 2$ such that*

$$W(t,x) \le a|x|^\nu + d \quad \text{for } (t,x)\in[0,T]\times\mathbb{R}^N;$$

(W6) *There exist $\alpha, \beta > 0$, $\mu \ge 2$ and $\mu > \nu - \gamma$ such that*

$$(\nabla W(t,x),x) - 2W(t,x) \ge \alpha|x|^\mu - \beta \quad \text{for } (t,x)\in[0,T]\times\mathbb{R}^N;$$

Then system (1.1) has a nontrivial T-periodic solution.

If $F(t,x)$ is symmetric in x, i.e. F satisfies

(F2) $F(t,-x) = F(t,x)$ for $(t,x)\in[0,T]\times\mathbb{R}^N$;

then we can obtain the following better result by using the Symmetric Mountain Pass Theorem.

Theorem 1.3. *Assume that F, K and W satisfy* (F1), (F2), (K1′), (K2′), (W1′), (W4′) *and* (W6). *Then system (1.1) has an unbounded sequence of T-periodic solutions.*

2. Proofs of theorems

Let

$$H_T^1 = \left\{u : [0,T]\to\mathbb{R}^N \mid u \text{ is absolutely continuous},\ u(0) = u(T),\ \dot{u}\in L^2([0,T],\mathbb{R}^N)\right\}.$$

Then H_T^1 is a Hilbert space with the norm defined by

$$\|u\| = \left(\int_0^T\left[|\dot{u}(t)|^2 + |u(t)|^2\right]\mathrm{d}t\right)^{\frac{1}{2}}$$

for $u\in H_T^1$. Let $I : H_T^1\to\mathbb{R}$ be defined by

$$I(u) = \int_0^T\left[\frac{1}{2}|\dot{u}(t)|^2 + K(t,u(t)) - W(t,u(t))\right]\mathrm{d}t. \tag{2.1}$$

Then $I \in C^1(H_T^1, \mathbb{R})$ and one can easily check that

$$\langle I'(u), v\rangle = \int_0^T [(\dot{u}(t), \dot{v}(t)) + (\nabla K(t, u(t)), v(t)) - (\nabla W(t, u(t)), v(t))]\,dt \tag{2.2}$$

for $u, v \in H_T^1$. It is well known that the T-periodic solutions of system (1.1) correspond to the critical points of I.

We will obtain the critical points of I by using the Mountain Pass Theorem and the Symmetric Mountain Pass Theorem. Since the minimax characterisation provides the critical value, it is important for what follows. Therefore, we state these theorems precisely.

Lemma 2.1 (*[24]*). *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfies the (PS) condition. If I satisfies the following conditions:*

(i) $I(0) = 0$;

(ii) *There exist constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha$;*

(iii) *There exists $e \in E \setminus \bar{B}_\rho(0)$ such that $I(e) \leq 0$, then I possesses a critical value $c \geq \alpha$ given by*

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

where $B_\rho(0)$ is an open ball in E of radius ρ centered at 0, and

$$\Gamma = \{g \in C([0, 1], E) : g(0) = 0, g(1) = e\}.$$

Lemma 2.2 (*[24]*). *Let E be a real Banach space, I is even and $I \in C^1(E, \mathbb{R})$ satisfies the (PS) condition. If I satisfies* (i), (ii) *of Lemma 2.1 and the following condition:*

(iii′) *For each finite dimensional subspace $E' \subset E$, there is $r = r(E')$ such that $I(u) \leq 0$ for $u \in E' \setminus B_r(0)$, where $B_r(0)$ is an open ball in E of radius r centered at 0.*

Then I possesses an unbounded sequence of critical values.

Remark 2.1. As shown in [2], a deformation lemma can be proved with condition (C) replacing the usual (PS) condition, and it turns out that Lemmas 2.1 and 2.2 hold true under condition (C). We say I satisfies condition (C), i.e., for every sequence $\{u_n\} \subset H_T^1$, $\{u_n\}$ has a convergent subsequence if $I(u_n)$ is bounded and $(1 + \|u_n\|)\|I'(u_n)\| \to 0$ as $n \to \infty$.

Lemma 2.3 (*[25]*). *For all $u \in H_T^1$,*

$$\|u\|_\infty \leq C_0\|u\| = \sqrt{\frac{1+\sqrt{1+T^2}}{2T}}\|u\|. \tag{2.3}$$

Proof of Theorem 1.1. By (W1′) and (W4), there exist constants $0 < \varepsilon < 1/2$, $p > 2$ and $C_1 > 1/C_0^2T$ such that

$$W(t, x) \leq (b - \varepsilon)|x|^2 + C_1|x|^p. \tag{2.4}$$

It is clear that $I(0) = 0$. We first show that I satisfies condition (C). Assume that $\{u_n\} \subset H_T^1$ is a (C) sequence of I, that is, $\{I(u_n)\}$ is bounded and $(1 + \|u_n\|)\|I'(u_n)\| \to 0$ as $n \to \infty$. Then it follows from (2.1), (2.2) and (K2′) that

$$\begin{aligned} C_2 &\geq 2I(u_n) - \langle I'(u_n), u_n\rangle \\ &= \int_0^T [(\nabla W(t, u_n(t)), u_n(t)) - 2W(t, u_n(t))]\,dt + \int_0^T [2K(t, u_n(t)) - (\nabla K(t, u_n(t)), u_n(t))]\,dt \\ &\geq \int_0^T [(\nabla W(t, u_n(t)), u_n(t)) - 2W(t, u_n(t))]\,dt. \end{aligned} \tag{2.5}$$

Now we prove $\{u_n\}$ is bounded by contradiction. If $\{u_n\}$ is unbounded, without loss of generality, we may assume that

$$\|u_n\| \to \infty \quad \text{as } n \to \infty. \tag{2.6}$$

Put $z_n = u_n/\|u_n\|$, we have $\|z_n\| = 1$. Going to a subsequence if necessary, we may assume that $z_n \rightharpoonup z$ weakly in H_T^1, and so $z_n \to z$ strongly in $C([0, T], \mathbb{R}^N)$. By (2.1), (K1′) and (W4), we have

$$\begin{aligned} I(u_n) &= \int_0^T \left[\frac{1}{2}|\dot{u}_n(t)|^2 + K(t, u_n(t)) - W(t, u_n(t))\right] dt \\ &\geq \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2 dt - a\int_0^T |u_n(t)|^2 dt - dT \\ &= \frac{1}{2}\|u_n\|^2 - \left(\frac{1}{2} + a\right)\int_0^T |u_n(t)|^2 dt - dT. \end{aligned} \tag{2.7}$$

Therefore, one has

$$\frac{I(u_n)}{\|u_n\|^2} \geq \frac{1}{2} - \left(\frac{1}{2}+a\right)\int_0^T |z_n(t)|^2 \mathrm{d}t - \frac{dT}{\|u_n\|^2}.$$

Passing to the limit in the above inequality, by using the fact that $\{I(u_n)\}$ is bounded and $\{z_n(t)\}$ converges uniformly to $z(t)$ on $[0, T]$, we obtain

$$\left(\frac{1}{2}+a\right)\int_0^T |z(t)|^2 \mathrm{d}t \geq \frac{1}{2},$$

which implies that $z \neq 0$. Set $E = \{t \in [0, T] : \lim_{|x|\to\infty}[(\nabla W(t, x), x) - 2W(t, x)] = +\infty\}$ and $E_1 = \{t \in [0, T] : z(t) \neq 0\} \bigcap E$. Then it follows from (W3′) that $\operatorname{meas}(E_1) > 0$ and

$$\lim_{n\to\infty} |u_n(t)| = +\infty \quad \text{for } t \in E_1. \tag{2.8}$$

Set $f_n(t) = (\nabla W(t, u_n(t)), u_n(t)) - 2W(t, u_n(t))$. Then by (2.8) and the definition of E_1, we have

$$\lim_{n\to\infty} f_n(t) = +\infty \quad \text{for } t \in E_1. \tag{2.9}$$

By (2.9) and Lemma 1 in [17], there exists a subset E_2 of E_1 with $\operatorname{meas}(E_2) > 0$ such that

$$\lim_{n\to\infty} f_n(t) = +\infty \quad \text{uniformly for } t \in E_2. \tag{2.10}$$

By (2.10), (W3′) and by using Fatou's lemma, we have

$$\begin{aligned}
&\liminf_{n\to\infty} \int_0^T [(\nabla W(t, u_n(t)), u_n(t)) - 2W(t, u_n(t))]\,\mathrm{d}t \\
&\quad = \liminf_{n\to\infty}\left\{\int_{E_2} [(\nabla W(t, u_n(t)), u_n(t)) - 2W(t, u_n(t))]\,\mathrm{d}t + \int_{[0,T]\setminus E_2} [(\nabla W(t, u_n(t)), u_n(t)) - 2W(t, u_n(t))]\,\mathrm{d}t\right\} \\
&\quad \geq \liminf_{n\to\infty} \int_{E_2} f_n(t)\mathrm{d}t + \int_{[0,T]\setminus E_2} g(t)\mathrm{d}t \\
&\quad \geq \int_{E_2} \liminf_{n\to\infty} f_n(t)\mathrm{d}t + \int_{[0,T]\setminus E_2} g(t)\mathrm{d}t \\
&\quad = +\infty,
\end{aligned} \tag{2.11}$$

which is a contradiction to (2.5). In a similar way to Proposition 4.3 in [12], we can prove that $\{u_n\}$ has a convergent subsequence. Thus, condition (C) holds.

We now show that there exist constants $\rho, \alpha > 0$ such that I satisfies the assumption (ii) of Lemma 2.1 with these constants. Let

$$\delta = \left(\frac{2\varepsilon}{pTC_0^2C_1}\right)^{1/(p-2)}. \tag{2.12}$$

Then $0 < \delta < 1$. If $\|u\| = \delta/C_0 := \rho$, then it follows from (2.3) that $|u(t)| \leq \delta < 1$ for $t \in [0, T]$. Set

$$\alpha = \frac{(p-2)\varepsilon}{p}\rho^2. \tag{2.13}$$

Hence, from (2.1), (2.3), (2.4), (2.12), (2.13) and (K1′), we have

$$\begin{aligned}
I(u) &= \int_0^T \left[\frac{1}{2}|\dot{u}(t)|^2 + K(t, u(t)) - W(t, u(t))\right]\mathrm{d}t \\
&\geq \frac{1}{2}\int_0^T |\dot{u}(t)|^2\mathrm{d}t + b\int_0^T |u(t)|^\gamma\,\mathrm{d}t - (b-\varepsilon)\int_0^T |u(t)|^2\mathrm{d}t - C_1\int_0^T |u(t)|^p\mathrm{d}t \\
&\geq \frac{1}{2}\int_0^T |\dot{u}(t)|^2\mathrm{d}t + \varepsilon\int_0^T |u(t)|^2\mathrm{d}t - C_1\int_0^T |u(t)|^p\mathrm{d}t \\
&\geq \varepsilon\left(\int_0^T |\dot{u}(t)|^2\mathrm{d}t + \int_0^T |u(t)|^2\mathrm{d}t\right) - C_1T\|u\|_\infty^p \\
&\geq \varepsilon\|u\|^2 - C_1TC_0^p\|u\|^p
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(p-2)\varepsilon}{p}\rho^2 \\
&= \alpha.
\end{aligned} \tag{2.14}
$$

(2.14) shows that $\|u\| = \rho$ implies that $I(u) \geq \alpha$.

Finally, it remains to show that I satisfies the assumption (iii) of Lemma 2.1. Set $\varphi(s) = s^{-2}W(t, sx_0)$ for $s > 0$. Then it follows from (W3′) that

$$
\varphi'(s) = s^{-3}[-2W(t, sx_0) + (\nabla W(t, sx_0), sx_0)] \geq s^{-3}g(t) \quad \text{for } t \in [0, T], s > 0.
$$

Integrating the above from 1 to $\zeta > 1$, we obtain

$$
W(t, \zeta x_0) \geq \zeta^2 W(t, x_0) + \frac{g(t)}{2}(\zeta^2 - 1) \quad \text{for } t \in [0, T], \zeta > 1. \tag{2.15}
$$

By (K2′), it is easy to show that

$$
K(t, \zeta x_0) \leq \zeta^2 K(t, x_0) \quad \text{for } t \in [0, T], \zeta > 1. \tag{2.16}
$$

From (2.1), (2.15), (2.16) and (W5), we have

$$
\begin{aligned}
I(\zeta x_0) &= \int_0^T [K(t, \zeta x_0) - W(t, \zeta x_0)]\,\mathrm{d}t \\
&\leq \zeta^2 \int_0^T \left[K(t, x_0) - W(t, x_0) - \frac{g(t)}{2}\right]\mathrm{d}t + \frac{1}{2}\int_0^T g(t)\mathrm{d}t \\
&\leq 0 \quad \text{for large } \zeta > 1.
\end{aligned} \tag{2.17}
$$

Choose $\sigma > 1$ such that $\sqrt{T}|\sigma x_0| > \rho$ and $I(\sigma x_0) \leq 0$. Let $e = \sigma x_0$. Then $e \in H_T^1$, $\|e\| = \sqrt{T}|\sigma x_0| > \rho$ and $I(e) \leq 0$. By Lemma 2.1, I possesses a critical value $c \geq \alpha$. Hence, there exists $u \in H_T^1$ such that

$$
I(u) = c \quad \text{and} \quad I'(u) = 0. \tag{2.18}
$$

Then function u is a desired nontrivial T-periodic solution of (1.1). The proof is complete. □

Proof of Corollary 1.1. Set

$$
\varepsilon = -\frac{1}{4T}\int_0^T \left[\max_{|x|=1} K(t, x) - \frac{V_\infty(t)}{2}\right]\mathrm{d}t. \tag{2.19}
$$

Then it follows from (W2′) that $\varepsilon > 0$ and there exists a constant $R > 0$ such that

$$
|\nabla W(t, x) - V_\infty(t)x| \leq \varepsilon|x| \quad \text{for } t \in [0, T],\ |x| \geq R. \tag{2.20}
$$

For any $x \in \mathbb{R}^N \setminus \{0\}$, let $x_* = Rx/|x|$. Then it follows from (2.20) that for all $t \in [0, T]$ and $x \in \mathbb{R}^N$ with $|x| > R$

$$
\begin{aligned}
W(t, x) - \frac{V_\infty(t)}{2}|x|^2 &= W(t, x_*) - \frac{V_\infty(t)}{2}|x_*|^2 + \int_0^1 (\nabla W(t, x_* + s(x - x_*)) - V_\infty(t)(x_* + s(x - x_*)), x - x_*)\mathrm{d}s \\
&\leq W(t, x_*) - \frac{R^2 V_\infty(t)}{2} + \varepsilon\int_0^1 |x_* + s(x - x_*)||x - x_*|\mathrm{d}s \\
&\leq W(t, x_*) - \frac{R^2 V_\infty(t)}{2} + 2\varepsilon|x|^2,
\end{aligned}
$$

which implies

$$
W(t, x) \leq \left(\frac{V_\infty(t)}{2} + 2\varepsilon\right)|x|^2 + W(t, x_*) - \frac{R^2 V_\infty(t)}{2}, \quad t \in [0, T],\ |x| > R. \tag{2.21}
$$

This shows that condition (W4) holds. Similarly, we have

$$
W(t, x) \geq \left(\frac{V_\infty(t)}{2} - 2\varepsilon\right)|x|^2 + W(t, x_*) - \frac{R^2 V_\infty(t)}{2}, \quad t \in [0, T],\ |x| > R. \tag{2.22}
$$

By (K2′), it is easy to show that

$$
K(t, x) \leq K\left(t, \frac{x}{|x|}\right)|x|^2 \quad \text{for } t \in [0, T],\ |x| > 1. \tag{2.23}
$$

Choose $x_0 \in \mathbb{R}^N$ such that $|x_0| > R + 1$ and

$$2\varepsilon T|x_0|^2 + \int_0^T \left[\frac{g(t)}{2} + \min_{|x|=R} W(t,x) - \frac{R^2 V_\infty(t)}{2}\right] \mathrm{d}t > 0. \tag{2.24}$$

Then it follows from (2.19), (2.22), (2.23), (2.24) and (W3′) that

$$\begin{aligned}
&\int_0^T \left[K(t,x_0) - W(t,x_0) - \frac{g(t)}{2}\right] \mathrm{d}t \\
&\le |x_0|^2 \int_0^T \left[K\left(t, \frac{x_0}{|x_0|}\right) - \frac{V_\infty(t)}{2} + 2\varepsilon\right] \mathrm{d}t - \int_0^T \left[\frac{g(t)}{2} + W(t,x_*) - \frac{R^2 V_\infty(t)}{2}\right] \mathrm{d}t \\
&\le |x_0|^2 \int_0^T \left[\max_{|x|=1} K(t,x) - \frac{V_\infty(t)}{2} + 2\varepsilon\right] \mathrm{d}t - \int_0^T \left[\frac{g(t)}{2} + \min_{|x|=R} W(t,x) - \frac{R^2 V_\infty(t)}{2}\right] \mathrm{d}t \\
&= -2\varepsilon T|x_0|^2 - \int_0^T \left[\frac{g(t)}{2} + \min_{|x|=R} W(t,x) - \frac{R^2 V_\infty(t)}{2}\right] \mathrm{d}t \\
&< 0.
\end{aligned}$$

This shows that condition (W5) also holds. By Theorem 1.1, the conclusion of Corollary 1.1 is true. The proof is complete. □

Proof of Theorem 1.2. We first show that I satisfies condition (C). Assume that $\{u_n\}_{n\in\mathbb{N}} \subset H_T^1$ is a (C) sequence of I, that is, $\{I(u_n)\}$ is bounded and $(1 + \|u_n\|)\|I'(u_n)\| \to 0$ as $n \to \infty$. Then it follows from (2.1), (2.2), (K2′) and (W6) that

$$\begin{aligned}
C_3 &\ge 2I(u_n) - \langle I'(u_n), u_n\rangle \\
&= \int_0^T [(\nabla W(t,u_n(t)), u_n(t)) - 2W(t,u_n(t))] \mathrm{d}t + \int_0^T [2K(t,u_n(t)) - (\nabla K(t,u_n(t)), u_n(t))] \mathrm{d}t \\
&\ge \int_0^T [(\nabla W(t,u_n(t)), u_n(t)) - 2W(t,u_n(t))] \mathrm{d}t \\
&\ge \int_0^T [\alpha|u_n(t)|^\mu - \beta] \mathrm{d}t.
\end{aligned} \tag{2.25}$$

From (2.1), (2.3), (2.25) and (W4′), we have

$$\begin{aligned}
\frac{1}{2}\int_0^T \left[|\dot{u}_n(t)|^2 + 2K(t,u_n(t))\right] \mathrm{d}t &= I(u_n) + \int_0^T W(t,u_n(t))\mathrm{d}t \\
&\le C_4 + dT + a\int_0^T |u_n(t)|^\nu \mathrm{d}t \\
&\le C_4 + dT + a\|u_n\|_\infty^{\nu-\mu} \int_0^T |u_n(t)|^\mu \mathrm{d}t \\
&\le C_4 + dT + \frac{aC_0^{\nu-\mu}(C_3 + \beta T)}{\alpha}\|u_n\|^{\nu-\mu}.
\end{aligned} \tag{2.26}$$

On the other hand, it follows from (K1′) and (2.3) that

$$\begin{aligned}
\frac{1}{2}\int_0^T \left[|\dot{u}_n(t)|^2 + 2K(t,u_n(t))\right] \mathrm{d}t &\ge \frac{1}{2}\int_0^T \left[|\dot{u}_n(t)|^2 + 2b|u_n(t)|^\gamma\right] \mathrm{d}t \\
&\ge \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2\mathrm{d}t + b(C_0\|u_n\|)^{\gamma-2}\int_0^T |u_n(t)|^2\mathrm{d}t \\
&\ge \min\left\{\frac{1}{2}, b(C_0\|u_n\|)^{\gamma-2}\right\}\left(\int_0^T |\dot{u}_n(t)|^2\mathrm{d}t + \int_0^T |u_n(t)|^2\mathrm{d}t\right) \\
&= \min\left\{\frac{1}{2}, b(C_0\|u_n\|)^{\gamma-2}\right\}\|u_n\|^2 \\
&= \min\left\{\frac{1}{2}\|u_n\|^2, bC_0^{\gamma-2}\|u_n\|^\gamma\right\}.
\end{aligned} \tag{2.27}$$

Combining (2.26) with (2.27), we obtain

$$\min\left\{\frac{1}{2}\|u_n\|^2, bC_0^{\gamma-2}\|u_n\|^{\gamma}\right\} \le C_4 + dT + \frac{aC_0^{\nu-\mu}(C_3+\beta T)}{\alpha}\|u_n\|^{\nu-\mu}. \tag{2.28}$$

Since $\nu-\mu<\gamma<2$, it follows that $\|u_n\|$ is bounded. In a similar way to Proposition 4.3 in [12], we can prove that $\{u_n\}$ has a convergent subsequence in H_T^1. Hence, I satisfies condition (C).

It is obvious that $I(0)=0$ and so conditions (i) of Lemma 2.1 holds. By (W1′) and (W4′), (2.4) still holds. Hence, condition (ii) of Lemma 2.1 can be proved as in the proof of Theorem 1.1. The proof of condition (iii) of Lemma 2.1 see the proof of condition (iii′) in the next theorem. By Lemma 2.1, I possesses a critical value $c\ge\alpha$. Hence, there exists $u\in H_T^1$ such that

$$I(u)=c \quad \text{and} \quad I'(u)=0. \tag{2.29}$$

Then function u is a desired nontrivial T-periodic solution of (1.1). The proof is complete. □

Proof of Theorem 1.3. (F2) implies that I is even. By Lemma 2.2 and the proof of Theorem 1.2, it suffices to prove that I satisfies (iii′) of Lemma 2.2. Let E' be a finite dimensional subspace of H_T^1. Since all the norms of a finite dimensional normed space are equivalent, so there is a constant $c>0$ such that

$$c\|u\| \le \|u\|_\mu = \left(\int_0^T |u(t)|^\mu \mathrm{d}t\right)^{1/\mu} \quad \text{for } u\in E'. \tag{2.30}$$

Set $\varphi(s)=s^{-2}W(t,sx)$ for $s>0$. Then it follows from (W6) that

$$\begin{aligned}\varphi'(s) &= s^{-3}[-2W(t,sx)+(\nabla W(t,sx),sx)]\\ &\ge \alpha s^{\mu-3}|x|^\mu - \beta s^{-3} \quad \text{for } (t,x)\in[0,T]\times\mathbb{R}^N \text{ and } s>0.\end{aligned}$$

Integrating the above from 1 to $\zeta>1$, we obtain

$$W(t,\zeta x)\ge \zeta^2 W(t,x) + \frac{\alpha}{\mu-2}\left(\zeta^\mu-\zeta^2\right)|x|^\mu - \frac{\beta}{2}\left(\zeta^2-1\right) \quad \text{for } (t,x)\in[0,T]\times\mathbb{R}^N \text{ and } \zeta>1, \tag{2.31}$$

where

$$\frac{1}{\mu-2}\left(\zeta^\mu-\zeta^2\right) = \lim_{s\to 2}\frac{\zeta^s-\zeta^2}{s-2} = \zeta^2\ln\zeta, \tag{2.32}$$

if $\mu=2$. By (K2′), it is easy to show that

$$K(t,\zeta x)\le \zeta^2 K(t,x) \quad \text{for } (t,x)\in[0,T]\times\mathbb{R}^N \text{ and } \zeta>1. \tag{2.33}$$

Set $\Theta=\{u\in E' : \|u\|=1\}$. From (2.3), (2.30), (2.31) and (2.33), we have for $u\in\Theta$ and $\zeta>1$

$$\begin{aligned}I(\zeta u) &= \frac{\zeta^2}{2}\int_0^T |\dot u(t)|^2\mathrm{d}t + \int_0^T K(t,\zeta u(t))\mathrm{d}t - \int_0^T W(t,\zeta u(t))\mathrm{d}t\\ &\le \frac{\zeta^2}{2}\left[\int_0^T |\dot u(t)|^2\mathrm{d}t + 2\int_0^T K(t,u(t))\mathrm{d}t - 2\int_0^T W(t,u(t))\mathrm{d}t\right]\\ &\quad - \frac{\alpha}{\mu-2}\left(\zeta^\mu-\zeta^2\right)\int_0^T |u(t)|^\mu\mathrm{d}t + \frac{\beta}{2}\left(\zeta^2-1\right)T\\ &\le \frac{\zeta^2}{2}\left\{\int_0^T |\dot u(t)|^2\mathrm{d}t + 2\int_0^T \max_{|x|\le C_0}[K(t,x)-W(t,x)]\mathrm{d}t\right\} - \frac{\alpha}{\mu-2}\left(\zeta^\mu-\zeta^2\right)\int_0^T |u(t)|^\mu\mathrm{d}t + \frac{\beta}{2}\left(\zeta^2-1\right)T\\ &\le \frac{\zeta^2}{2}\left(\|u\|^2+C_6\right) - \frac{\alpha}{\mu-2}\left(\zeta^\mu-\zeta^2\right)\|u\|_\mu^\mu + \frac{\beta}{2}\left(\zeta^2-1\right)T\\ &\le \frac{\zeta^2}{2}\left(\|u\|^2+C_6\right) - \frac{\alpha}{\mu-2}\left(\zeta^\mu-\zeta^2\right)c^\mu\|u\|^\mu + \frac{\beta}{2}\left(\zeta^2-1\right)T.\end{aligned} \tag{2.34}$$

Since $\mu\ge 2$, it follows from (2.32) and (2.34) that there is $\zeta_0=\sigma(c,\Theta)=\zeta_0(E')>1$ such that

$$I(\zeta u)<0 \quad \text{for } u\in\Theta \text{ and } \zeta\ge\zeta_0.$$

That is

$$I(u)<0 \quad \text{for } u\in E' \text{ and } \|u\|\ge\zeta_0.$$

This shows that (iii′) of Lemma 2.2 holds. By Lemma 2.2, I possesses an unbounded sequence $\{d_n\}_{n=1}^{\infty}$ of critical values with $d_n = I(u_n)$, where u_n is such that $I'(u_n) = 0$ for $n = 1, 2, \ldots$. Set

$$a_1 = \max\{K(t,x) : t \in [0,T], x \in \mathbb{R}^N, |x| = 1\},$$
$$a_2 = \max\{K(t,x) : t \in [0,T], x \in \mathbb{R}^N, |x| \le 1\},$$

and

$$a_3 = \min\{W(t,x) : t \in [0,T], x \in \mathbb{R}^N, |x| = 1\}.$$

Then by (K2′) and the fact that $K \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $0 \le a_1 \le a_2 < \infty$ and

$$K(t,x) \le a_1|x|^2 + a_2 \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}^N. \tag{2.35}$$

On the other hand, it follows from (2.31) and (2.32) that

$$\begin{aligned} W(t,x) &\ge W(t,x/|x|)|x|^2 + \frac{\alpha}{\mu-2}\left(|x|^{\mu} - |x|^2\right) - \frac{\beta}{2}\left(|x|^2 - 1\right) \\ &\ge a_3|x|^2 + \frac{\alpha}{\mu-2}\left(|x|^{\mu} - |x|^2\right) - \frac{\beta}{2}\left(|x|^2 - 1\right) \\ &\ge C_7 \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}^N \text{ and } |x| \ge 1. \end{aligned} \tag{2.36}$$

Set $\Lambda_n = \{t \in [0,T] : |u_n(t)| \le 1\}$ for $n = 1, 2, \ldots$. Then

$$\begin{aligned} \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2 \mathrm{d}t &= I(u_n) + \int_0^T [W(t,u_n(t)) - K(t,u_n(t))]\,\mathrm{d}t \\ &= d_n + \int_{\Lambda_n} W(t,u_n(t))\mathrm{d}t + \int_{[0,T]\setminus\Lambda_n} W(t,u_n(t))\mathrm{d}t - \int_0^T K(t,u_n(t))\mathrm{d}t \\ &\ge d_n + C_8 - a_1\int_0^T |u_n(t)|^2\mathrm{d}t, \end{aligned}$$

which implies that

$$d_n \le \left(\frac{1}{2} + a_1\right)\|u_n\|^2 - C_8, \quad n = 1, 2, \ldots. \tag{2.37}$$

Since $d_n \to \infty$ as $n \to \infty$, it follows from (2.37) that $\|u_n\| \to \infty$ as $n \to \infty$. The proof is complete. □

3. Examples

In this section, we give two examples to illustrate our results.

Example 3.1. Let

$$W(t,x) = a(t)|x|^2\left(1 - \frac{1}{\ln(e+|x|)}\right), \qquad K(t,x) = b|x|^{\gamma} + b_1(t)|x|^{\varrho}, \tag{3.1}$$

where $b > 0$, $a, b_1 \in C(\mathbb{R}, [0,+\infty))$, $1 < \gamma < \varrho \le 2$, and $a(t+T) = a(t)$, $b_1(t+T) = b_1(t)$. It is easy to verify that K and W satisfy conditions (F1), (K1′), (K2′), (W1) and (W4). Since

$$(\nabla W(t,x), x) - 2W(t,x) = \frac{a(t)|x|^3}{(e+|x|)[\ln(e+|x|)]^2}.$$

Then W satisfies (W3′) with $g(t) \equiv 0$ if

$$\operatorname{meas}\{t \in [0,T] : a(t) = 0\} = 0. \tag{3.2}$$

In addition,

$$\begin{aligned} \int_0^T \left[K(t,x) - W(t,x) - \frac{g(t)}{2}\right]\mathrm{d}t &= \int_0^T \left[b|x|^{\gamma} + b_1(t)|x|^{\varrho} - a(t)|x|^2\left(1 - \frac{1}{\ln(e+|x|)}\right)\right]\mathrm{d}t \\ &= bT|x|^{\gamma} + |x|^{\varrho}\int_0^T b_1(t)\mathrm{d}t - |x|^2\left(1 - \frac{1}{\ln(e+|x|)}\right)\int_0^T a(t)\mathrm{d}t. \end{aligned}$$

The above shows that there exists $x_0 \in \mathbb{R}^N$ such that (W5) holds if

$$\int_0^T a(t)\mathrm{d}t > \int_0^T b_1(t)\mathrm{d}t. \tag{3.3}$$

By Theorem 1.1, if (3.2) and (3.3) hold, then system (1.1) with K and W as in (3.1) has a nontrivial T-periodic solution.

Example 3.2. Let

$$W(t,x) = a(t)|x|^2 \ln(1+|x|), \qquad K(t,x) = b|x|^\gamma + b_1(t)|x|^\varrho, \tag{3.4}$$

where $b > 0$, $a, b_1 \in C(\mathbb{R}, [0, +\infty))$, $1 < \gamma < \varrho \le 2$, and $a(t+T) = a(t)$, $b_1(t+T) = b_1(t)$. It is easy to verify that K and W satisfy conditions (F1), (F2), (K1′), (K2′) and (W1). Since

$$(\nabla W(t,x), x) - 2W(t,x) = \frac{a(t)|x|^3}{1+|x|}.$$

Then W satisfies (W4′) and (W6) with $\mu = 2$ and $\nu = 3$ if $a(t) > 0$. By Theorem 1.3, system (1.1) with K and W as in (3.4) has an unbounded sequence of T-periodic solutions.

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